

O'Connell's process as a vicious Brownian motion

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Abstract

Vicious Brownian motion is a diffusion scaling limit of Fisher's vicious walk model, which is a system of Brownian particles in one dimension such that if two of them meet they kill each other. We consider the vicious Brownian motion conditioned never to collide with each other, and call it the noncolliding Brownian motion. This conditional diffusion process is equivalent to the eigenvalue process of a Hermitian-matrix-valued Brownian motion studied by Dyson. Recently O'Connell introduced a generalization of the noncolliding Brownian motion by using the eigenfunctions (the Whittaker functions) of the quantum Toda lattice in order to analyze a directed polymer model in 1+1 dimensions. We consider a system of one-dimensional Brownian motions with a long-ranged killing term as a generalization of the vicious Brownian motion and construct the O'Connell process as a conditional process of the killing Brownian motions to survive forever.

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I. INTRODUCTION

Vicious walk model introduced by Fisher [1] is the system of one-dimensional random walkers such that, if neighboring walkers meet, they kill each other. Though the model looks sinister, what we are interested in is to evaluate the probability that for a finite time-interval any neighboring pair of vicious walkers do not meet and thus all walkers survive; in other words, the probability that the peace is kept [2–4]. If we take appropriate continuum limit (the diffusion scaling limit), we obtain “vicious Brownian motion” (vicious BM) [5, 6]. Assume that the number of particles of vicious BM is $N \geq 2$ and write the positions as $x_j, 1 \leq j \leq N$. Then the configuration space of them conditioned never to collide is

$$\mathbb{W}_N = \{\mathbf{x} = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : x_1 < x_2 < \dots < x_N\}, \quad (1)$$

which is called the Weyl chamber of type A_{N-1} in the representation theory [7]. The boundaries of this region $\partial\mathbb{W}_N$ in the N -dimensional real space \mathbb{R}^N consists of the hyperplanes $x_j = x_{j+1}, 1 \leq j \leq N-1$, each of which corresponds to occurrence of collision of the j -th and $(j+1)$ -th particles in the vicious BM. If we regard \mathbf{x} as a position vector of the N -dimensional BM within \mathbb{W}_N and $\partial\mathbb{W}_N$ as an absorbing boundary such that when a particle hit the boundary it is immediately absorbed, the system is identified with the absorbing BM in \mathbb{W}_N . The harmonic function, $\Delta h_N(\mathbf{x}) \equiv \sum_{j=1}^N \partial^2 h_N(\mathbf{x}) / \partial x_j^2 = 0, \mathbf{x} \in \mathbb{W}_N$, satisfying the Dirichlet boundary condition $h_N(\mathbf{x}) = 0, \mathbf{x} \in \partial\mathbb{W}_N$, is uniquely determined up to a constant factor as

$$h_N(\mathbf{x}) = \prod_{1 \leq j < k \leq N} (x_k - x_j), \quad \mathbf{x} \in \mathbb{W}_N, \quad (2)$$

which is equal to the Vandermonde determinant $\det_{1 \leq j, k \leq N} [x_j^{k-1}]$. Then we can show that the survival probability up to time $t \geq 0$ of N -particle system of vicious BM starting from an initial configuration $\mathbf{x} \in \mathbb{W}_N$ has the asymptotics

$$h_N\left(\frac{\mathbf{x}}{\sqrt{t}}\right) = t^{-N(N-1)/4} h_N(\mathbf{x}) \quad \text{in} \quad \frac{|\mathbf{x}|}{\sqrt{t}} \rightarrow 0, \quad (3)$$

where $|\mathbf{x}| = \sqrt{x_1^2 + \dots + x_N^2}$ (see, for instance, [8]).

As a matter of course, the survival probability decreases in time for any $\mathbf{x} \in \mathbb{W}_N$. The result (3) implies, however, the decay is slow in the sense that it is not exponential but follows the power-law in time. This observation has led us to study the system of BMs conditioned never to collide with each other, which we call the *noncolliding BM*. The fundamental

properties of this process are the following [8]. (i) Let the transition probability density of a single BM be $p(t, y|x) = e^{-(y-x)^2/2t}/\sqrt{2\pi t}$. Then the transition probability density of the absorbing BM in \mathbb{W}_N from a configuration $\mathbf{x} \in \mathbb{W}_N$ to $\mathbf{y} \in \mathbb{W}_N$ in time interval $t \geq 0$ is given by the Karlin-McGregor determinant [9]

$$q_N(t, \mathbf{y}|\mathbf{x}) = \det_{1 \leq j, k \leq N} [p(t, y_j|x_k)], \quad (4)$$

or equivalently [10] given by the Harish-Chandra-Izykson-Zuber integral [11, 12]

$$q_N(t, \mathbf{y}|\mathbf{x}) = \frac{t^{-N^2/2}}{c_N} h_N(\mathbf{x}) h_N(\mathbf{y}) \int_{\mathbb{U}(N)} e^{-\text{Tr}(\Lambda_{\mathbf{x}} - U^\dagger \Lambda_{\mathbf{y}} U)^2/2t} dU, \quad (5)$$

where dU is the Haar measure of the space of unitary matrices $\mathbb{U}(N)$ normalized as $\int_{\mathbb{U}(N)} dU = 1$, $\Lambda_{\mathbf{x}} = \text{diag}(x_1, x_2, \dots, x_N)$, $\Lambda_{\mathbf{y}} = \text{diag}(y_1, y_2, \dots, y_N)$, and $c_N = (2\pi)^{N/2} \prod_{j=1}^N \Gamma(j)$ with the gamma function $\Gamma(j)$. (ii) The transition probability density of the noncolliding BM, $p_N(t, \mathbf{y}|\mathbf{x})$, is then given by the harmonic transform of $q_N(t, \mathbf{y}|\mathbf{x})$ with (2) in the sense of Doob [13],

$$p_N(t, \mathbf{y}|\mathbf{x}) = \frac{h_N(\mathbf{y})}{h_N(\mathbf{x})} q_N(t, \mathbf{y}|\mathbf{x}), \quad \mathbf{x}, \mathbf{y} \in \mathbb{W}_N, \quad t \geq 0. \quad (6)$$

(iii) If we regard (6) as a function of t and initial configuration \mathbf{x} , it is a solution of the following backward Kolmogorov equation

$$\begin{aligned} \frac{\partial}{\partial t} u(t, \mathbf{x}) &= \frac{1}{2} \Delta u(t, \mathbf{x}) + \nabla \log h_N(\mathbf{x}) \cdot \nabla u(t, \mathbf{x}) \\ &= \frac{1}{2} \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} u(t, \mathbf{x}) + \sum_{1 \leq j \leq N} \sum_{1 \leq k \leq N: k \neq j} \frac{1}{x_j - x_k} \frac{\partial}{\partial x_j} u(t, \mathbf{x}) \end{aligned} \quad (7)$$

under the initial condition $u(0, \mathbf{x}) = \delta(\mathbf{x} - \mathbf{y}) \equiv \prod_{j=1}^N \delta(x_j - y_j)$.

The direct consequence of the above facts is the following (see, for example, [14]). Let $\mathbf{X}(t) = (X_1(t), \dots, X_N(t))$ be the N -particle system of the noncolliding BM. Then it solves the system of stochastic differential equations (SDEs)

$$dX_j(t) = dB_j(t) + \sum_{1 \leq k \leq N: k \neq j} \frac{dt}{X_j(t) - X_k(t)}, \quad 1 \leq j \leq N, \quad t \geq 0, \quad (8)$$

where $B_j(t)$, $1 \leq j \leq N$, $t \geq 0$, are independent one-dimensional standard BMs. Eq.(8) is nothing but the system of SDEs for Dyson's BM model with the parameter $\beta = 2$ [15] studied in the random matrix theory [16, 17], which we will simply call *the Dyson model*

in this paper. That is, the noncolliding BM is equivalent to the eigenvalue process of the Hermitian-matrix-valued BM [10, 18].

Recently O’Connell introduced a family of diffusion processes of N particles in one dimension, $\mathbf{Z}^{\vec{\mu}}(t) = (Z_1^{\vec{\mu}}(t), \dots, Z_N^{\vec{\mu}}(t)), t \geq 0$ with a parameter given by an N -dimensional real vector $\vec{\mu} = (\mu_1, \mu_2, \dots, \mu_N) \in \mathbb{R}^N$, in which the j -th particle has a constant drift μ_j , $1 \leq j \leq N$ [19]. It is an extension of the Dyson model (8) in the sense that, if we consider the scaled process $\varepsilon \mathbf{Z}^0(t/\varepsilon^2), t \geq 0$ with $\varepsilon > 0$ for the case $\vec{\mu} = 0$ and take the limit $\varepsilon \rightarrow 0$, then the limit process is equivalent to $\mathbf{X}(t), t \geq 0$. He showed that the process $\mathbf{Z}^{\vec{\mu}}(t), t \geq 0$ is associated with the quantum Toda lattice with the Hamiltonian [20–22]

$$\mathcal{H}_N = -\frac{1}{2} \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + \sum_{j=1}^{N-1} e^{-(x_{j+1} - x_j)}. \quad (9)$$

The O’Connell process is very rich in mathematics connecting with quantum integrable systems, representation theory of Lie groups/algebras, the Whittaker functions, theory of intertwining relations of Markov processes, and so on. He discussed the importance of his process to study a model of 1+1 dimensional directed polymers in random environment with finite temperature [19].

The purpose of the present paper is to discuss the O’Connell process as a generalized version of vicious BM with appropriate conditions at least for the special case $\vec{\mu} = 0$. (See [23–38] for other generalizations and recent topics of vicious BM and noncolliding BM. We note that interesting connections between random growth models and the Toda lattice Hamiltonian is discussed in [39].)

The paper is organized as follows. In Sec.II through the Feynman-Kac formula, we introduce a system of Brownian particles with the killing term which is in the same form as the potential term in the quantum Toda lattice Hamiltonian (9) and discuss it as a generalization of the vicious BM. In Sec.III the transition probability density $Q_N(t, \mathbf{y}|\mathbf{x})$ of the N -particle system of killing BMs is expressed as an integral of a product of eigenfunctions of the quantum Toda lattice over the Sklyanin measure. Then asymptotics of $Q_N(t, \mathbf{y}|\mathbf{x})$ in $t \rightarrow \infty$ is estimated (Lemma 1). In Sec.IV we introduce a drift $\vec{\mu}$ in our N -particle system of killing BMs and define the transition probability density of the killing BMs conditioned to survive up to time $0 < T < \infty$. By taking the double limits $T \rightarrow \infty$ and $\vec{\mu} \rightarrow 0$, we obtain the transition probability density $P_N(t, \mathbf{y}|\mathbf{x})$ for the killing BMs with $\vec{\mu} = 0$ conditioned to survive forever. The main theorem is given there (Theorem 2), by which the

equivalence between the present conditional process and the O'Connell process with $\vec{\mu} = 0$ is concluded. We discuss a one-dimensional diffusion process studied by Matsumoto and Yor [40, 41] in Sec.V as a motion of relative coordinate in the $N = 2$ case of our process. The Matsumoto-Yor process with $\mu = 0$ is realized as a one-dimensional killing BM conditioned to survive forever. In Sec.VI we discuss some distributions obtained by setting the special initial conditions. Section VII is devoted to summary and concluding remarks. Appendix A is given for proving an asymptotics used in Sec.V. Some details of the $N = 2$ case of the O'Connell process are given in Appendix B.

II. QUANTUM TODA LATTICE AND FEYNMAN-KAC FORMULA

Let $N \in \{2, 3, \dots\}$. Consider the eigenvalue problem of the quantum Toda lattice Hamiltonian (9),

$$\mathcal{H}_N \Psi_\gamma(\mathbf{x}) = \gamma \Psi_\gamma(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^N. \quad (10)$$

For $\vec{\lambda} = (\lambda_1, \dots, \lambda_N) \in \mathbb{C}^N$, the eigenfunctions of (10) with eigenvalues

$$\gamma = -\frac{1}{2} \sum_{j=1}^N \lambda_j^2 \quad (11)$$

have been extensively studied [20–22], which are expressed by $\psi_{\vec{\lambda}}^{(N)}(\mathbf{x})$ in the present paper. Let \mathbf{T} denote a triangular array with size N , $\mathbf{T} = (T_{k,j}, 1 \leq j \leq k \leq N)$. We consider that the $N(N-1)/2$ elements $T_{k,j}$ of \mathbf{T} are independent variables and introduce a function of them as

$$\begin{aligned} \mathcal{F}_{\vec{\lambda}}^{(N)}(\mathbf{T}) = & \sum_{k=1}^N \lambda_k \left(\sum_{j=1}^k T_{k,j} - \sum_{j=1}^{k-1} T_{k-1,j} \right) \\ & - \sum_{k=1}^{N-1} \sum_{j=1}^k \left\{ e^{-(T_{k,j} - T_{k+1,j})} + e^{-(T_{k+1,j+1} - T_{k,j})} \right\}, \end{aligned} \quad (12)$$

which depends on $\vec{\lambda} = (\lambda_1, \dots, \lambda_N)$. For a given $\mathbf{x} \in \mathbb{R}^N$, let $\Gamma_N(\mathbf{x})$ be the space of all real triangular arrays \mathbf{T} with size N conditioned

$$T_{N,j} = x_j, \quad 1 \leq j \leq N. \quad (13)$$

We write the integral of a function f of \mathbf{T} over $\Gamma_N(\mathbf{x})$ as

$$\int_{\Gamma_N(\mathbf{x})} f(\mathbf{T}) d\mathbf{T} \equiv \prod_{k=1}^N \prod_{j=1}^k \int_{-\infty}^{\infty} dT_{k,j} f(\mathbf{T}) \prod_{\ell=1}^N \delta(T_{N,\ell} - x_\ell). \quad (14)$$

Then the integral representation of $\psi_{\vec{\lambda}}^{(N)}(\mathbf{x})$ is given by

$$\psi_{\vec{\lambda}}^{(N)}(\mathbf{x}) = \int_{\Gamma_N(\mathbf{x})} e^{\mathcal{F}_{\vec{\lambda}}^{(N)}(\mathbf{T})} d\mathbf{T}. \quad (15)$$

This multivariate function is a version of Whittaker function (see [19] and references therein).

As a stochastic version of the Schrödinger equation of the quantum Toda lattice (obtained by performing the Wick rotation in the Schrödinger equation), we consider the following diffusion equation

$$\frac{\partial}{\partial t} u(t, \mathbf{x}) = \mathcal{L}_N u(t, \mathbf{x}) \quad (16)$$

with the infinitesimal generator of the process

$$\mathcal{L}_N \equiv -\mathcal{H}_N = \frac{1}{2} \Delta - V_N(\mathbf{x}), \quad (17)$$

where

$$V_N(\mathbf{x}) = \sum_{j=1}^{N-1} e^{-(x_{j+1} - x_j)}. \quad (18)$$

If we follow the method of separation of variables by setting $u(t, \mathbf{x}) = T(t)\Psi_{\gamma}(\mathbf{x})$, (16) is decomposed into the equations

$$\frac{dT(t)}{dt} = -\gamma T(t)$$

and (10). Then we can conclude that for any $\vec{\lambda} \in \mathbb{C}^N$,

$$\exp\left(\frac{t}{2} \sum_{j=1}^N \lambda_j^2\right) \psi_{\vec{\lambda}}^{(N)}(\mathbf{x}) \quad (19)$$

solves the diffusion equation (16).

In the context of quantum mechanics, the function $V_N(\mathbf{x})$ given by (18) plays, as a matter of course, a role of potential energy. Then the quantum system prefers the state $x_{j+1} > x_j$ to the state $x_{j+1} < x_j$, $1 \leq j \leq N-1$, since the former has lower energy than the latter. On the other hand, in the context of stochastic calculus, $-V_N(\mathbf{x})$ term in the infinitesimal generator of the process (17) acts as a *killing term*. We consider N independent one-dimensional standard BMs starting from 0, $B_j(t)$, $1 \leq j \leq N$, and for $\mathbf{x} \in \mathbb{R}^N$ set $\mathbf{B}^{\mathbf{x}}(t) = \mathbf{x} + \mathbf{B}(t)$, where each element $B_j^{x_j}(t) = x_j + B_j(t)$ is a one-dimensional standard BM starting from x_j , $1 \leq j \leq N$. Then the Feynman-Kac formula (see, for instance, [14]) implies that the function

$$Q_N(t, \mathbf{y}|\mathbf{x}) = \mathbb{E} \left[\mathbf{1}(\mathbf{B}^{\mathbf{y}}(t) = \mathbf{x}) \exp \left\{ - \int_0^t V_N(\mathbf{B}^{\mathbf{y}}(s)) ds \right\} \right] \quad (20)$$

solves the diffusion equation (16) under the initial condition

$$Q_N(0, \mathbf{y}|\mathbf{x}) = \delta(\mathbf{x} - \mathbf{y}), \quad (21)$$

where $\mathbb{E}[\cdot]$ denotes the expectation over all realizations of N -dimensional Brownian paths, $\{\mathbf{B}^{\mathbf{y}}(s) : 0 \leq s \leq t\}$, starting from \mathbf{y} , and $\mathbf{1}(\omega)$ is the indicator function of the event ω ; $\mathbf{1}(\omega) = 1$ if ω is satisfied, $\mathbf{1}(\omega) = 0$ otherwise. The function $Q_N(t, \mathbf{y}|\mathbf{x})$ is the transition probability density of the process (16) from a configuration \mathbf{x} to a configuration \mathbf{y} in time interval $t \geq 0$. In the Feynman-Kac formula (20), we consider a collection of all paths of BM in \mathbb{R}^N starting from \mathbf{y} to \mathbf{x} . (Though the time direction is backward, it is irrelevant in calculation, since BM is time-reversible.) The point of this formula is the following. In order to give the transition probability density $Q_N(t, \mathbf{y}|\mathbf{x})$, we have to put a weight

$$\begin{aligned} w_N &= \exp \left\{ - \int_0^t V_N(\mathbf{B}^{\mathbf{y}}(s)) ds \right\} \\ &= \exp \left\{ - \sum_{j=1}^{N-1} \int_0^t e^{-(B_{j+1}^{y_{j+1}}(s) - B_j^{y_j}(s))} ds \right\} \end{aligned} \quad (22)$$

to each realization of path of the N -dimensional BM and take a summation over all realizations of paths. It is obvious that w_N takes a real value in $[0, 1]$. Then this summation of weighted paths (a path integral) can be identified with a statistical-ensemble average of Brownian paths, in which each path is included in the ensemble with probability w_N and is deleted with probability $1 - w_N$. Deletion of an N -dimensional Brownian path is interpreted as an event that the N -dimensional BM is killed in the time interval $[0, t]$. The weight w_N is then regarded as the probability that the particle in \mathbb{R}^N survives up to time t . (See Corollary 4.5 and explanation given below it in Chapter 4 of [14] for the equivalence of the Feynman-Kac formula to Brownian motion with killing of particles.) Eq.(22) gives the dependence of the survival probability on the realization of path $\{\mathbf{B}^{\mathbf{y}}(s), 0 \leq s \leq t\}$. If the N -tuples of Brownian paths are “well-ordered” in the spatio-temporal plane, $B_1^{y_1}(s) < B_2^{y_2}(s) < \dots < B_N^{y_N}(s), 0 \leq s \leq t$, and moreover $B_{j+1}^{y_{j+1}}(s) \gg B_j^{y_j}(s), 1 \leq j \leq N-1, 0 \leq s \leq t$, w_N is large, while for a particle on the path $\{\mathbf{B}^{\mathbf{y}}(s), 0 \leq s \leq t\}$ in which $B_{j+1}^{y_{j+1}}(s) < B_j^{y_j}(s), 1 \leq j \leq N$ for some $s \in [0, t]$, w_N is small.

If we introduce a parameter $\varepsilon > 0$, then we can see that

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \exp \left\{ - \sum_{j=1}^{N-1} \int_0^t e^{-(B_{j+1}^{y_{j+1}}(s) - B_j^{y_j}(s))/\varepsilon} ds \right\} \\
&= \mathbf{1} \left(B_1^{y_1}(s), \dots, B_N^{y_N}(s) \text{ do not collide during } [0, t] \right) \\
&= \mathbf{1} \left(\mathbf{B}^y(s) \in \mathbb{W}_N, 0 \leq \forall s \leq t \right).
\end{aligned} \tag{23}$$

In this sense, the process (16) with (17) and (18) is an N -particle system of killing BMs, which can be regarded as an extension of the absorbing BM in \mathbb{W}_N . In the next section, we explain how to express the transition probability density given by the Feynman-Kac formula (20) as a superposition of the Toda lattice eigenfunctions (19).

Remark. If we consider the present process not as an N -dimensional BM in \mathbb{R}^N but as an N -particle system of one-dimensional BMs, (20) gives the transition probability density in the case that mutual killing of particles does not occur at all in time duration t , since \mathbf{x} and \mathbf{y} are both N -particle configurations, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$. In order to discuss processes, in which mutual killing of particles actually occurs and total number of particles decreases in time, we have to specify the way how to choose pair of particles which are annihilated; *e.g.* the pair $(j, j+1)$ attaining $\min\{B_{k+1}^{y_{k+1}}(t) - B_k^{y_k}(t)\}$ is chosen. Note that in the original vicious BM, colliding pairs of particles are pair annihilated. In the present paper, however, we are interested in the process conditioned that all N particles survive.

III. TRANSITION PROBABILITY DENSITY AND ITS LONG-TERM ASYMPTOTICS

The problem which is discussed here is how to determine the function $g_{\vec{\lambda}}(\mathbf{y})$ of $\vec{\lambda} \in \mathbb{C}^N, \mathbf{y} \in R^N$ and a subset Σ of \mathbb{C}^N such that the integral of (19)

$$\int_{\Sigma} \exp \left(\frac{t}{2} \sum_{j=1}^N \lambda_j^2 \right) \psi_{\vec{\lambda}}^{(N)}(\mathbf{x}) g_{\vec{\lambda}}(\mathbf{y}) d\vec{\lambda} \tag{24}$$

is equal to $Q_N(t, \mathbf{y}|\mathbf{x})$ given by (20). This problem is solved by applying the theory of the Sklyanin measure [42] defined by,

$$s_N(\vec{\lambda}) d\vec{\lambda} \equiv \frac{1}{(2\pi i)^N N!} \prod_{1 \leq j < k \leq N} \left\{ (\lambda_k - \lambda_j) \frac{\sin \pi(\lambda_j - \lambda_k)}{\pi} \right\} \prod_{\ell=1}^N d\lambda_{\ell} \tag{25}$$

for $\vec{\lambda} \in \Sigma = (i\mathbb{R})^N$, where $i = \sqrt{-1}$. As a multi-dimensional extension of the fact that Macdonald's functions of imaginary order $\lambda \in i\mathbb{R}$, $K_\lambda(e^{-x})$, make complete basis of a suitable set of functions with respect to the measure $s_1(\lambda)d\lambda = (i/\pi)^2 \lambda \sin(\pi\lambda)d\lambda$ (see p.131 of [43] and Sec.V in the present paper);

$$\frac{i}{\pi^2} \int_{-i\infty}^{i\infty} K_\lambda(e^{-x}) K_\lambda(e^{-y}) \lambda \sin(\pi\lambda) d\lambda = \delta(x - y), \quad (26)$$

$x, y \in \mathbb{R}$, the following is valid [20, 21],

$$\int_{(i\mathbb{R})^N} \psi_{\vec{\lambda}}^{(N)}(\mathbf{x}) \psi_{-\vec{\lambda}}^{(N)}(\mathbf{y}) s_N(\vec{\lambda}) d\vec{\lambda} = \delta(\mathbf{x} - \mathbf{y}), \quad (27)$$

$\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$. Note that if $\vec{\lambda} \in (i\mathbb{R})^N$ then $\psi_{-\vec{\lambda}}^{(N)}(\mathbf{y})$ is the complex conjugate of $\psi_{\vec{\lambda}}^{(N)}(\mathbf{y})$.

Since the transition probability density (20) is a unique solution of the diffusion equation (16) satisfying the initial condition (21), the following is concluded;

$$Q_N(t, \mathbf{y}|\mathbf{x}) = \int_{(i\mathbb{R})^N} e^{t \sum_{j=1}^N \lambda_j^2/2} \psi_{\vec{\lambda}}^{(N)}(\mathbf{x}) \psi_{-\vec{\lambda}}^{(N)}(\mathbf{y}) s_N(\vec{\lambda}) d\vec{\lambda}. \quad (28)$$

It should be noted that Lemma 4.6 in [20] implies

$$\int_{\mathbb{R}^N} \psi_{-\vec{\tau}}^{(N)}(\mathbf{x}) \psi_{\vec{\lambda}}^{(N)}(\mathbf{x}) d\mathbf{x} = \frac{1}{s_N(\vec{\lambda})} \frac{1}{N!} \sum_{\sigma \in \mathcal{S}_N} \delta(\sigma(\vec{\lambda}) - \vec{\tau}) \quad (29)$$

for $\vec{\lambda}, \vec{\tau} \in (i\mathbb{R})^N$, where \mathcal{S}_N is a set of all permutations of N indices and $\sigma(\vec{\lambda}) \equiv (\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(N)})$ for $\sigma \in \mathcal{S}_N$. The orthogonality (29) guarantees the Chapman-Kolmogorov equation for the transition probability density

$$\int_{\mathbb{R}^N} Q_N(t, \mathbf{z}|\mathbf{y}) Q_N(s, \mathbf{y}|\mathbf{x}) d\mathbf{y} = Q_N(t+s, \mathbf{z}|\mathbf{x}), \quad \mathbf{x}, \mathbf{z} \in \mathbb{R}^N \quad (30)$$

for $0 \leq s, t < \infty$. As a matter of course, (20) should satisfy it by the Markov property of BMs.

If we change the integral variables in (28) with (25) as $\lambda_j \mapsto \nu_j$ by $\lambda_j \sqrt{t/2} = i\nu_j$, $1 \leq j \leq N$, we obtain the following expression for $Q_N(t, \mathbf{y}|\mathbf{x})$,

$$Q_N(t, \mathbf{y}|\mathbf{x}) = \frac{2^{N^2/2}}{(2\pi)^N N!} t^{-N^2/2} \psi_0^{(N)}(\mathbf{x}) \psi_0^{(N)}(\mathbf{y}) \\ \times \int_{\mathbb{R}^N} e^{-|\vec{v}|^2} \prod_{1 \leq j < k \leq N} \left[(\nu_k - \nu_j) \frac{\sinh\{\pi\sqrt{2/t}(\nu_k - \nu_j)\}}{\pi\sqrt{2/t}} \right] \frac{\psi_{i\sqrt{2/t}\vec{v}}^{(N)}(\mathbf{x}) \psi_{-i\sqrt{2/t}\vec{v}}^{(N)}(\mathbf{y})}{\psi_0^{(N)}(\mathbf{x}) \psi_0^{(N)}(\mathbf{y})} d\vec{v}, \quad (31)$$

$\mathbf{x}, \mathbf{y} \in \mathbb{R}^N, t \geq 0$. This integral expression should be compared with the Schur function expansion of (4) [10],

$$q_N(t, \mathbf{y}|\mathbf{x}) = \frac{t^{-N^2/2}}{(2\pi)^{N/2}} h_N(\mathbf{x}) h_N(\mathbf{y}) \times \sum_{\nu: \ell(\nu) \leq N} \prod_{j=1}^N \frac{1}{(\nu_j + N - j)!} \exp\left(-\frac{|\mathbf{x}|^2 + |\mathbf{y}|^2}{2t}\right) s_\nu(\mathbf{x}/\sqrt{t}) s_\nu(\mathbf{y}/\sqrt{t}), \quad (32)$$

where $\{\nu = (\nu_1, \nu_2, \dots) : \nu_1 \geq \nu_2 \geq \dots\}$ are partitions of integers, $\ell(\nu)$ denotes the length of ν , and $s_\nu(\mathbf{x})$ is the Schur function [7].

We can prove the following asymptotics of the transition probability density.

Lemma 1 Let

$$\alpha_N = \frac{N^2}{2} \quad (33)$$

and $C_N = \prod_{n=1}^N \Gamma(n)/(2\pi)^{N/2}$. Then

$$\lim_{t \rightarrow \infty} \frac{t^{\alpha_N}}{C_N} Q_N(t, \mathbf{y}|\mathbf{x}) = \psi_0^{(N)}(\mathbf{x}) \psi_0^{(N)}(\mathbf{y}) \quad (34)$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$.

Proof. In the limit $t \rightarrow \infty$, (31) will behave as

$$Q_N(t, \mathbf{y}|\mathbf{x}) \simeq \frac{2^{N^2/2}}{(2\pi)^N N!} t^{-N^2/2} \psi_0^{(N)}(\mathbf{x}) \psi_0^{(N)}(\mathbf{y}) \int_{\mathbb{R}^N} e^{-|\vec{v}|^2} \prod_{1 \leq j < k \leq N} (\nu_k - \nu_j)^2 d\vec{v}. \quad (35)$$

The integral is a version of the Selberg integral (a special case with $\gamma = 1$ and $a = 1$ in Eq. (17.6.7) of [16]),

$$\int_{\mathbb{R}^N} e^{-|\vec{v}|^2} \prod_{1 \leq j < k \leq N} (\nu_k - \nu_j)^2 = (2\pi)^{N/2} 2^{-N^2/2} \prod_{n=1}^N n!. \quad (36)$$

Then the lemma is obtained. ■

IV. O'CONNELL PROCESS AS AN N -PARTICLE SYSTEM OF KILLING BROWNIAN MOTIONS CONDITIONED TO SURVIVE FOREVER

Let $\vec{\mu} = (\mu_1, \dots, \mu_N) \in \mathbb{R}^N$ and introduce a drift term in the diffusion equation

$$\frac{\partial}{\partial t} u^{\vec{\mu}}(t, \mathbf{x}) = \mathcal{L}_N^{\vec{\mu}} u^{\vec{\mu}}(t, \mathbf{x}) \quad (37)$$

with

$$\begin{aligned}\mathcal{L}_N^{\vec{\mu}} &= \mathcal{L}_N - \vec{\mu} \cdot \nabla \\ &= \mathcal{L}_N - \sum_{j=1}^N \mu_j \frac{\partial}{\partial x_j}.\end{aligned}\tag{38}$$

It is easy to confirm that if $u(t, \mathbf{x})$ solves (16), then

$$\begin{aligned}u^{\vec{\mu}}(t, \mathbf{x}) &= \exp\left(-\frac{t}{2}|\vec{\mu}|^2 + \vec{\mu} \cdot \mathbf{x}\right) u(t, \mathbf{x}) \\ &= \exp\left(-\frac{t}{2} \sum_{j=1}^N \mu_j^2 + \sum_{j=1}^N \mu_j x_j\right) u(t, \mathbf{x})\end{aligned}\tag{39}$$

solves (37). The formula (39) is called the drift transformation from $u(t, \mathbf{x})$ to $u^{\vec{\mu}}(t, \mathbf{x})$ [14]. Then we can prove that the transition probability density for the process with drift $\vec{\mu}$ is given by

$$Q_N^{\vec{\mu}}(t, \mathbf{y}|\mathbf{x}) = \exp\left\{-\frac{t}{2}|\vec{\mu}|^2 + \vec{\mu} \cdot (\mathbf{x} - \mathbf{y})\right\} Q_N(t, \mathbf{y}|\mathbf{x})\tag{40}$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N, t \geq 0, \vec{\mu} \in \mathbb{R}^N$.

For $\mathbf{x} \in \mathbb{R}^N, 0 \leq T < \infty$, let

$$\mathcal{N}_N^{\vec{\mu}}(T, \mathbf{x}) = \int_{\mathbb{R}^N} Q_N^{\vec{\mu}}(T, \mathbf{y}|\mathbf{x}) d\mathbf{y},\tag{41}$$

which gives the probability that all N particles survive up to time T in the process of killing BMs with drift $\vec{\mu}$.

Now we consider the killing BM with drift $\vec{\mu}$ conditioned to survive up to time T . Assume that $0 \leq s \leq t \leq T$ and let $P_{N,T}^{\vec{\mu}}(s, \mathbf{x}; t, \mathbf{y})$ denote the transition probability density from a configuration \mathbf{x} at time s to a configuration \mathbf{y} at time t of this conditional process as illustrated by Fig.1. By the Markov property of BMs, we treat the paths only after time s for $P_{N,T}^{\vec{\mu}}(s, \mathbf{x}; t, \mathbf{y})$. Since configurations at times s and t are both specified to be \mathbf{x} and \mathbf{y} , but configuration at the final time T is arbitrary, $P_{N,T}^{\vec{\mu}}(s, \mathbf{x}; t, \mathbf{y})$ should be proportional to a product of $Q_N^{\vec{\mu}}(t-s, \mathbf{y}|\mathbf{x})$ and $\mathcal{N}_N^{\vec{\mu}}(T-t, \mathbf{y})$. The product should be divided by $\mathcal{N}_N^{\vec{\mu}}(T-s, \mathbf{x})$ to give $P_{N,T}^{\vec{\mu}}(s, \mathbf{x}; t, \mathbf{y})$, since $\int_{\mathbb{R}^N} P_{N,T}^{\vec{\mu}}(s, \mathbf{x}; t, \mathbf{y}) d\mathbf{y} = 1$ should hold for any $\mathbf{x} \in \mathbb{R}^N$. Here we can see that by the Chapman-Kolmogorov equation (30), $\int_{\mathbb{R}^N} \mathcal{N}_N^{\vec{\mu}}(T-t, \mathbf{y}) Q_N^{\vec{\mu}}(t-s, \mathbf{y}|\mathbf{x}) d\mathbf{y} = \mathcal{N}_N^{\vec{\mu}}(T-s, \mathbf{x})$. That is, we have the formula

$$P_{N,T}^{\vec{\mu}}(s, \mathbf{x}; t, \mathbf{y}) = \frac{\mathcal{N}_N^{\vec{\mu}}(T-t, \mathbf{y}) Q_N^{\vec{\mu}}(t-s, \mathbf{y}|\mathbf{x})}{\mathcal{N}_N^{\vec{\mu}}(T-s, \mathbf{x})}.\tag{42}$$

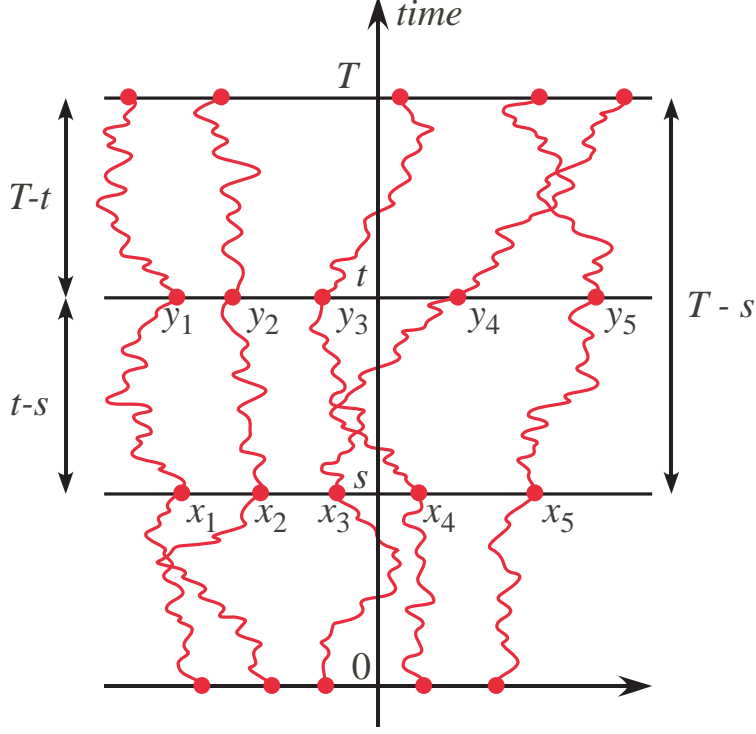


FIG. 1: An illustration of the paths in the case $N = 5$ and $\vec{\mu} = 0$ such that all particles survive up to time T , in which particle configurations at times s and t , $0 \leq s \leq t \leq T$, are specified to be $\mathbf{x} = (x_1, x_2, \dots, x_5)$ and $\mathbf{y} = (y_1, y_2, \dots, y_5)$, respectively. By the Markov property of BMs, the behavior of paths after time s is independent of that before time s . The total survival probability of the process during time interval $[s, T]$ is $\mathcal{N}_N^{\vec{\mu}}(T-s, \mathbf{x})$ under the condition that the configuration at time s is \mathbf{x} . If we fix the configuration at time t to be \mathbf{y} , $s \leq t \leq T$, the survival probability density is given by a product $\mathcal{N}_N^{\vec{\mu}}(T-t, \mathbf{y})Q_N^{\vec{\mu}}(t-s, \mathbf{y}|\mathbf{x})$, since the events before time t and after time t are independent by the Markov property of BMs. As shown by (42), the ratio of them gives the transition probability density $P_{N,T}^{\vec{\mu}}(s, \mathbf{x}; t, \mathbf{y})$.

By definition, this conditional process is a temporally inhomogeneous process, which is clarified by the fact that the transition probability density (42) is the function not only of the time difference $t-s$ but also of $T-s$ and $T-t$.

In order to obtain the temporally homogeneous process, we take the limit $T \rightarrow \infty$. Here we note that $\psi_0^{(N)}(\mathbf{y})$ is the eigenfunction of the quantum Toda lattice (9) with zero eigenvalue. If $y_{j+1} \gg y_j$ for all $1 \leq j \leq N-1$, the potential energy (18) becomes zero and then $\psi_0^{(N)}(\mathbf{y})$ behaves similar to the harmonic function $h_N(\mathbf{y})$. On the other hand, it is known

that $\psi_{\vec{\lambda}}^{(N)}(\mathbf{y}) \sim \exp\{-2e^{-(y_{j+1}-y_j)}\} \rightarrow 0$ as $y_{j+1}-y_j \rightarrow -\infty$, $1 \leq j \leq N-1$ for any $\vec{\lambda} \in (i\mathbb{R})^N$ [20]. Then if $\vec{\mu}$ is chosen so that $\vec{\mu} \in \mathbb{W}_N$, then the integral $\int_{\mathbb{R}^N} \psi_0^{(N)}(\mathbf{y}) \exp(-\vec{\mu} \cdot \mathbf{y}) d\mathbf{y}$ is finite. In such a case, by Lemma 1 and the drift transformation (40),

$$\lim_{T \rightarrow \infty} \frac{T^{\alpha_N}}{C_N} e^{|\vec{\mu}|^2 T/2} \mathcal{N}_N^{\vec{\mu}}(T-s, \mathbf{x}) = \exp(\vec{\mu} \cdot \mathbf{x}) \psi_0^{(N)}(\mathbf{x}) I_0^{\vec{\mu}}(N), \quad (43)$$

$0 < \forall s < \infty$, with

$$I_0^{\vec{\mu}}(N) = \int_{\mathbb{R}^N} \psi_0^{(N)}(\mathbf{y}) \exp(-\vec{\mu} \cdot \mathbf{y}) d\mathbf{y}, \quad \vec{\mu} \in \mathbb{W}_N. \quad (44)$$

Then we have

$$\begin{aligned} P_N^{\vec{\mu}}(t-s, \mathbf{y}|\mathbf{x}) &\equiv \lim_{T \rightarrow \infty} P_{N,T}^{\vec{\mu}}(s, \mathbf{x}; t, \mathbf{y}) \\ &= \exp\{-\vec{\mu} \cdot (\mathbf{x} - \mathbf{y})\} \frac{\psi_0^{(N)}(\mathbf{y})}{\psi_0^{(N)}(\mathbf{x})} Q_N^{\vec{\mu}}(t-s, \mathbf{y}|\mathbf{x}) \\ &= e^{-|\vec{\mu}|^2 t/2} \frac{\psi_0^{(N)}(\mathbf{y})}{\psi_0^{(N)}(\mathbf{x})} Q_N(t-s, \mathbf{y}|\mathbf{x}) \end{aligned} \quad (45)$$

for $0 \leq s \leq t < \infty$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$, $\vec{\mu} \in \mathbb{W}_N$.

Finally we take the limit $\vec{\mu} \rightarrow 0$, $\vec{\mu} \in \mathbb{W}_N$. The obtained transition probability density is given by

$$\begin{aligned} P_N(t, \mathbf{y}|\mathbf{x}) &\equiv \lim_{\vec{\mu} \rightarrow 0, \vec{\mu} \in \mathbb{W}_N} P_N^{\vec{\mu}}(t, \mathbf{y}|\mathbf{x}) \\ &= \frac{\psi_0^{(N)}(\mathbf{y})}{\psi_0^{(N)}(\mathbf{x})} Q_N(t, \mathbf{y}|\mathbf{x}), \end{aligned} \quad (46)$$

$\mathbf{x}, \mathbf{y} \in \mathbb{R}^N, t \geq 0$. It is an extension of (6), where the Vandermonde determinant $h_N(\mathbf{x})$ is replaced by the eigenfunction $\psi_0^{(N)}(\mathbf{x})$ with zero eigenvalue of the quantum Toda lattice. Now we state the main theorem of the present paper.

Theorem 2 The function (46), which is obtained as the transition probability density of the killing BM conditioned to survive forever, solves the following differential equation,

$$\begin{aligned} \frac{\partial}{\partial t} u(t, \mathbf{x}) &= \frac{1}{2} \Delta u(t, \mathbf{x}) + \nabla \log \psi_0^{(N)}(\mathbf{x}) \cdot \nabla u(t, \mathbf{x}) \\ &= \frac{1}{2} \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} u(t, \mathbf{x}) + \sum_{j=1}^N \frac{\partial \log \psi_0^{(N)}(\mathbf{x})}{\partial x_j} \frac{\partial}{\partial x_j} u(t, \mathbf{x}) \end{aligned} \quad (47)$$

under the initial condition

$$u(0, \mathbf{x}) = \delta(\mathbf{x} - \mathbf{y}). \quad (48)$$

Proof. By (46),

$$\begin{aligned}
\frac{\partial}{\partial x_j} P_N(t, \mathbf{y}|\mathbf{x}) &= -\frac{\psi_0^{(N)}(\mathbf{y})}{(\psi_0^{(N)}(\mathbf{x}))^2} \frac{\partial \psi_0^{(N)}(\mathbf{x})}{\partial x_j} Q_N(t, \mathbf{y}|\mathbf{x}) + \frac{\psi_0^{(N)}(\mathbf{y})}{\psi_0^{(N)}(\mathbf{x})} \frac{\partial}{\partial x_j} Q_N(t, \mathbf{y}|\mathbf{x}), \\
\frac{\partial^2}{\partial x_j^2} P_N(t, \mathbf{y}|\mathbf{x}) &= \frac{2\psi_0^{(N)}(\mathbf{y})}{(\psi_0^{(N)}(\mathbf{x}))^3} \left(\frac{\partial \psi_0^{(N)}(\mathbf{x})}{\partial x_j} \right)^2 Q_N(t, \mathbf{y}|\mathbf{x}) \\
&\quad - \frac{\psi_0^{(N)}(\mathbf{y})}{(\psi_0^{(N)}(\mathbf{x}))^2} \frac{\partial^2 \psi_0^{(N)}(\mathbf{x})}{\partial x_j^2} Q_N(t, \mathbf{y}|\mathbf{x}) - 2 \frac{\psi_0^{(N)}(\mathbf{y})}{(\psi_0^{(N)}(\mathbf{x}))^2} \frac{\partial \psi_0^{(N)}(\mathbf{x})}{\partial x_j} \frac{\partial}{\partial x_j} Q_N(t, \mathbf{y}|\mathbf{x}) \\
&\quad + \frac{\psi_0^{(N)}(\mathbf{y})}{\psi_0^{(N)}(\mathbf{x})} \frac{\partial^2}{\partial x_j^2} Q_N(t, \mathbf{y}|\mathbf{x}).
\end{aligned} \tag{49}$$

Since

$$\frac{\partial \log \psi_0^{(N)}(\mathbf{x})}{\partial x_j} = \frac{1}{\psi_0^{(N)}(\mathbf{x})} \frac{\partial \psi_0^{(N)}(\mathbf{x})}{\partial x_j},$$

RHS of (47) is equal to

$$-\frac{\psi_0^{(N)}(\mathbf{y})}{(\psi_0^{(N)}(\mathbf{x}))^2} Q_N(t, \mathbf{y}|\mathbf{x}) \frac{1}{2} \sum_{j=1}^N \frac{\partial^2 \psi_0^{(N)}(\mathbf{x})}{\partial x_j^2} + \frac{\psi_0^{(N)}(\mathbf{y})}{\psi_0^{(N)}(\mathbf{x})} \frac{1}{2} \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} Q_N(t, \mathbf{y}|\mathbf{x}). \tag{50}$$

Since $\psi_0^{(N)}(\mathbf{x})$ is the eigenfunction of (9) with zero eigenvalue,

$$\mathcal{H}_N \psi_0^{(N)}(\mathbf{x}) = -\frac{1}{2} \sum_{j=1}^N \frac{\partial^2 \psi_0^{(N)}(\mathbf{x})}{\partial x_j^2} + \sum_{j=1}^{N-1} e^{-(x_{j+1}-x_j)} \psi_0^{(N)}(\mathbf{x}) = 0, \tag{51}$$

and $Q_N(t, \mathbf{y}|\mathbf{x})$ satisfies the equation

$$\frac{\partial}{\partial t} Q_N(t, \mathbf{y}|\mathbf{x}) = \frac{1}{2} \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} Q_N(t, \mathbf{y}|\mathbf{x}) - \sum_{j=1}^{N-1} e^{-(x_{j+1}-x_j)} Q_N(t, \mathbf{y}|\mathbf{x}), \tag{52}$$

we can see that (50) is equal to $\partial P_N(t, \mathbf{y}|\mathbf{x})/\partial t$; that is, $P_N(t, \mathbf{y}|\mathbf{x})$ satisfies (47). Since $Q_N(0, \mathbf{y}|\mathbf{x}) = \delta(\mathbf{x} - \mathbf{y})$, (46) gives $P_N(0, \mathbf{y}|\mathbf{x}) = \delta(\mathbf{x} - \mathbf{y})$. The proof is then completed. ■

O'Connell [19] introduced a diffusion process in \mathbb{R}^N with the infinitesimal generator of the process

$$\mathcal{G}_N^{\vec{\mu}} = \frac{1}{2} \Delta + \nabla \log \psi_{\vec{\mu}}^{(N)}(\mathbf{x}) \cdot \nabla \tag{53}$$

with $\vec{\mu} \in \mathbb{R}^N$. Our theorem states that the special case with $\vec{\mu} = 0$ can be realized as a vicious BM, which is obtained above as a system of killing BMs conditioned to survive forever. Corresponding to the backward Kolmogorov equation (47), the SDEs of the process $\mathbf{Z}(t) = (Z_1(t), \dots, Z_N(t)), t \geq 0$ is then given by

$$dZ_j(t) = dB_j(t) + F_j^{(N)}(\mathbf{Z}(t))dt, \quad 1 \leq j \leq N, t \geq 0 \tag{54}$$

with

$$F_j^{(N)}(\mathbf{x}) = \frac{\partial \log \psi_0^{(N)}(\mathbf{x})}{\partial x_j}, \quad (55)$$

where $B_j(t)$, $1 \leq j \leq N$, $t \geq 0$ are independent one-dimensional BMs.

V. MATSUMOTO-YOR PROCESS AS A KILLING BROWNIAN MOTION CONDITIONED TO SURVIVE FOREVER

For $N = 2$, the eigenfunction (15) of the quantum Toda lattice is

$$\begin{aligned} \psi_{(\lambda_1, \lambda_2)}^{(2)}(x_1, x_2) &= \int_{\Gamma_2(x_1, x_2)} e^{\mathcal{F}_{(\lambda_1, \lambda_2)}^{(2)}(\mathbf{T})} d\mathbf{T} \\ &= \int_{-\infty}^{\infty} \exp \left[\lambda_1 T_{1,1} + \lambda_2 (x_1 + x_2 - T_{1,1}) - \{e^{-(T_{1,1}-x_1)} + e^{-(x_2-T_{1,1})}\} \right] dT_{1,1} \\ &= e^{\lambda_2(x_1+x_2)} \int_{-\infty}^{\infty} e^{(\lambda_1-\lambda_2)T_{1,1}} \exp \left[- \left(\frac{e^{x_1}}{e^{T_{1,1}}} + \frac{e^{T_{1,1}}}{e^{x_2}} \right) \right] dT_{1,1}, \end{aligned} \quad (56)$$

$(\lambda_1, \lambda_2) \in \mathbb{C}^2$, $(x_1, x_2) \in \mathbb{R}^2$. Change the integral variables $T_{1,1} \mapsto s$ by $e^{T_{1,1}} = e^{(x_1+x_2)/2} s$. Then

$$\psi_{(\lambda_1, \lambda_2)}^{(2)}(x_1, x_2) = e^{(\lambda_1+\lambda_2)(x_1+x_2)/2} \int_0^\infty s^{\lambda_1-\lambda_2-1} \exp \left[-e^{-(x_2-x_1)/2} \left(s + \frac{1}{s} \right) \right] ds. \quad (57)$$

Let $I_\nu(z)$ be the modified Bessel function of the first kind

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}, \quad |z| < \infty, \quad |\arg z| < \pi \quad (58)$$

and $K_\nu(z)$ be Macdonald's function [43]

$$K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}, \quad |\arg z| < \pi, \quad \text{for } \nu \neq 0, \pm 1, \pm 2, \dots, \quad (59)$$

and for integers $\nu = n$,

$$K_n(z) = \lim_{\nu \rightarrow n} K_\nu(z), \quad n = 0, \pm 1, \pm 2, \dots, \quad (60)$$

which are both analytic functions of z for all z in the complex plane cut along the negative real axis, and entire functions of ν . We can see that $I_\nu(z)$ and $K_\nu(z)$ are linearly independent solutions of the differential equation

$$\frac{d^2 w}{dz^2} + \frac{1}{z} \frac{dw}{dz} - \left(1 + \frac{\nu^2}{z^2} \right) w = 0. \quad (61)$$

For $x > 0$ and $\nu \geq 0$, $I_\nu(x)$ is a positive function which increases monotonically as $x \rightarrow \infty$, while $K_\nu(x)$ is a positive function which decreases monotonically as $x \rightarrow \infty$. Since $K_\nu(z)$ has the following integral representation

$$K_\nu(z) = \frac{1}{2} \int_0^\infty s^{\nu-1} \exp \left[-\frac{z}{2} \left(s + \frac{1}{s} \right) \right] ds, \quad (62)$$

(57) is written as

$$\psi_{(\lambda_1, \lambda_2)}^{(2)}(x_1, x_2) = 2e^{(\lambda_1 + \lambda_2)(x_1 + x_2)/2} K_{\lambda_1 - \lambda_2}(2e^{-(x_2 - x_1)/2}). \quad (63)$$

The infinitesimal generator of the process (53) is then given for $N = 2$ as

$$\begin{aligned} \mathcal{G}_2^{(\mu_1, \mu_2)} &= \frac{1}{2} \Delta + \sum_{j=1}^2 \frac{\partial}{\partial x_j} \log \psi_{\lambda_1, \lambda_2}^{(2)}(x_1, x_2) \frac{\partial}{\partial x_j} \\ &= \frac{1}{2} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) + \frac{1}{2} (\mu_1 + \mu_2) \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) \\ &\quad + \frac{K'_{\mu_1 - \mu_2}(2e^{-(x_2 - x_1)/2})}{K_{\mu_1 - \mu_2}(2e^{-(x_2 - x_1)/2})} e^{-(x_2 - x_1)/2} \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right), \end{aligned} \quad (64)$$

where $K'_\nu(z) \equiv dK_\nu(z)/dz$. If we change the variables $(x_1, x_2) \mapsto (\xi, \eta)$ by $\xi = (x_1 + x_2)/2$, $\eta = -(x_1 - x_2)/2 - \log 2$, (64) is decomposed into two parts, $\mathcal{G}_2^{(\mu_1, \mu_2)} = (\mathcal{G}_0^{\mu_1 + \mu_2} + \mathcal{G}_{\text{MY}}^{\mu_1 - \mu_2})/2$, where

$$\mathcal{G}_0^\nu = \frac{1}{2} \frac{d^2}{d\xi^2} + \nu \frac{d}{d\xi}, \quad (65)$$

$$\mathcal{G}_{\text{MY}}^\mu = \frac{1}{2} \frac{d^2}{d\eta^2} + \frac{d}{d\eta} \{ \log K_\mu(e^{-\eta}) \} \frac{d}{d\eta} = \frac{1}{2} \frac{d^2}{d\eta^2} - \frac{K'_\mu(e^{-\eta})}{K_\mu(e^{-\eta})} e^{-\eta} \frac{d}{d\eta}. \quad (66)$$

The former is the infinitesimal generator of the one-dimensional BM with a constant drift $\nu = \mu_1 + \mu_2$ and the latter is that of the diffusion process studied by Matsumoto and Yor with parameter $\mu = \mu_1 - \mu_2$ [40, 41]. It implies that, in the $N = 2$ case of the O'Connell process with parameter $\vec{\mu} = (\mu_1, \mu_2) \in \mathbb{R}^2$, the center of mass ξ behaves as (a time change $t \mapsto 2t$ of) a BM with a drift $\mu_1 + \mu_2$ and the relative coordinate η behaves as (a time change $t \mapsto 2t$ of) the Matsumoto-Yor process with parameter $\mu_1 - \mu_2$.

For $\mu \in \mathbb{R}$, let

$$\mathcal{L}^\mu = \frac{1}{2} \frac{d^2}{dx^2} - V(x) - \mu \frac{d}{dx} \quad (67)$$

with

$$V(x) = \frac{1}{2} e^{-2x}. \quad (68)$$

It is the infinitesimal generator of the one-dimensional drifted BM with a killing term $-V(x)$. The transition probability density is given by

$$\begin{aligned} Q^\mu(t, y|x) &= \mathbb{E} \left[\mathbf{1}(B^{\mu, y}(t) = x) \exp \left\{ -\frac{1}{2} \int_0^t e^{-2B^{\mu, y}(s)} ds \right\} \right] \\ &= e^{-\mu^2 t/2 + \mu(x-y)} \frac{i}{\pi^2} \int_{-i\infty}^{i\infty} e^{\lambda^2 t/2} K_\lambda(e^{-x}) K_\lambda(e^{-y}) \lambda \sin(\pi \lambda) d\lambda, \end{aligned} \quad (69)$$

where $B^{\mu, y}(t) = y + B(t) + \mu t$ with the one-dimensional standard BM, $B(t)$, starting from 0; $B(0) = 0$.

The survival probability up to time $T > 0$ of this one-dimensional killing BM with drift μ is given by

$$\mathcal{N}^\mu(T, x) = \int_{-\infty}^{\infty} Q^\mu(T, y|x) dy \quad (70)$$

for the initial position $x \in \mathbb{R}$. We can prove that, if $\mu > 0$, it has the long-term asymptotics,

$$\begin{aligned} \lim_{T \rightarrow \infty} \sqrt{\frac{\pi}{2}} T^{3/2} e^{\mu^2 T/2} \mathcal{N}^\mu(T, x) &= e^{\mu x} K_0(e^{-x}) \int_{-\infty}^{\infty} K_0(e^{-y}) e^{-\mu y} dy \\ &= 2^{\mu-2} (\Gamma(\mu/2))^2 e^{\mu x} K_0(e^{-y}). \end{aligned} \quad (71)$$

The proof of (71) is given in Appendix A. Then the transition probability density for this *one-dimensional killing BM conditioned to survive forever* is given in the limit $\mu \rightarrow 0, \mu > 0$ as

$$\begin{aligned} P(t, y|x) &= \lim_{\mu \rightarrow 0, \mu > 0} \lim_{T \rightarrow \infty} \frac{\mathcal{N}^\mu(T-t, y)}{\mathcal{N}^\mu(T, x)} Q^\mu(t, y|x) \\ &= \frac{K_0(e^{-y})}{K_0(e^{-x})} \frac{i}{\pi^2} \int_{-i\infty}^{i\infty} e^{\lambda^2 t/2} K_\lambda(e^{-x}) K_\lambda(e^{-y}) \lambda \sin(\pi \lambda) d\lambda, \end{aligned} \quad (72)$$

$x, y \in \mathbb{R}, t > 0$. It is easy to see that (72) satisfies the diffusion equation

$$\frac{\partial}{\partial t} u(t, x) = \mathcal{G}_{\text{MY}}^0 u(t, x), \quad t \geq 0, \quad (73)$$

under the initial condition $u(0, x) = \delta(x - y)$.

Matsumoto and Yor showed that the stochastic process

$$Z_{\text{MY}}^\mu(t) = \log \left\{ \int_0^t e^{2B^\mu(s)} ds \right\} - B^\mu(t), \quad t \geq 0 \quad (74)$$

with $B^\mu(t) \equiv B^{\mu, 0}(t) = B(t) + \mu t$ is a diffusion process, whose infinitesimal generator is given by (66) for any $\mu \in \mathbb{R}$. Here we have shown that, when $\mu = 0$, the Matsumoto-Yor process (74) can be constructed as a one-dimensional killing BM conditioned to survive forever.

See Appendix B for more detail on the relation between the Matsumoto-Yor process and the $N = 2$ case of the O'Connell process.

VI. SPECIAL INITIAL CONDITIONS

In this section, first we consider the one-dimensional diffusion process with the infinitesimal generator (67) with (68). Let $0 < T < \infty$. Then the transition probability density of the process conditioned to survive up to time T is given by

$$\begin{aligned} P_T^\mu(s, x; t, y) &= \frac{\mathcal{N}^\mu(T - t, y)}{\mathcal{N}^\mu(T - s, x)} Q^\mu(t - s, y|x) \\ &= \mathcal{N}^\mu(T - t, y) \frac{Q^\mu(t - s, y|x)}{\int_{-\infty}^{\infty} Q^\mu(T - s, z|x) dz}, \end{aligned} \quad (75)$$

$0 \leq s \leq t \leq T, x, y \in \mathbb{R}$, where Q^μ and \mathcal{N}^μ are given by (69) and (70), respectively. The asymptotics of $K_\lambda(e^{-x})$ in $x \rightarrow -\infty$ is independent of λ [43]

$$K_\lambda(e^{-x}) \simeq \sqrt{\frac{\pi}{2e^{-x}}} \exp(-e^{-x}) \quad \text{as } x \rightarrow -\infty. \quad (76)$$

Then if we define

$$P_T^\mu(t, y | -\infty) \equiv \lim_{x \rightarrow -\infty} P_T^\mu(0, x; t, y), \quad (77)$$

it is given by

$$P_T^\mu(t, y | -\infty) = \frac{\sqrt{2\pi}}{2^{\mu-2}(\Gamma(\mu/2))^2} T^{3/2} e^{\mu^2(T-t)/2 - \pi^2/T} \theta_{e^{-y}}(t) e^{-\mu y} \mathcal{N}(T - t, y), \quad (78)$$

where $\theta_r(t)$ is given by (A9) and (A10) in Appendix A [44]. Since $\mathcal{N}^\mu(0, y) = 1$ by definition, we obtain the distribution at time $t = T$,

$$P_T^\mu(T, y | -\infty) = c^\mu(T) \theta_{e^{-y}}(T) e^{-\mu y}, \quad y \in \mathbb{R}, \quad T > 0 \quad (79)$$

with $c^\mu(T) = \sqrt{2\pi} 2^{2-\mu} (\Gamma(\mu/2))^{-2} T^{3/2} e^{-\pi^2/T}$. On the other hand, by (71), if we take the temporally homogeneous limit $T \rightarrow \infty$ in (78), then we obtain the distribution

$$\begin{aligned} P^\mu(t, y | -\infty) &\equiv \lim_{T \rightarrow \infty} P_T^\mu(t, y | -\infty) \\ &= 2e^{-\mu^2 t/2} \theta_{e^{-y}}(t) K_0(e^{-y}), \quad y \in \mathbb{R}, \quad t > 0. \end{aligned} \quad (80)$$

Next we consider the N -particle system of the killing BMs with drift $\vec{\mu} \in \mathbb{W}_N$ conditioned to survive up to time $T, 0 < T < \infty$. The transition probability density is given by (42). Corresponding to (76), the asymptotics of $\psi_{\vec{\lambda}}^{(N)}(\mathbf{x})$ in $x_j \rightarrow -\infty, 1 \leq \forall j \leq N$ is independent of $\vec{\lambda}$ (see Remark 8.1 in [19]). Then we obtain

$$\begin{aligned} P_{N,T}^{\vec{\mu}}(t, \mathbf{y} | -\infty) &\equiv \lim_{x_j \rightarrow -\infty, 1 \leq j \leq N} P_{N,T}^{\vec{\mu}}(0, \mathbf{x}; t, \mathbf{y}) \\ &= \frac{e^{|\vec{\mu}|^2(T-t)/2}}{J^{\vec{\mu}}(N, T)} \Theta_N(t, \mathbf{y}) \exp(-\vec{\mu} \cdot \mathbf{y}) \mathcal{N}_N^{\vec{\mu}}(T - t, \mathbf{y}), \end{aligned} \quad (81)$$

where

$$J^{\vec{\mu}}(N, T) = \int_{(i\mathbb{R})^N} e^{T \sum_{j=1}^N \lambda_j^2/2} s_N(\vec{\lambda}) I_{-\vec{\lambda}}^{\vec{\mu}}(N) d\vec{\lambda} \quad (82)$$

with

$$I_{-\vec{\lambda}}^{\vec{\mu}}(N) = \int_{\mathbb{R}^N} \psi_{-\vec{\lambda}}^{(N)}(\mathbf{z}) \exp(-\vec{\mu} \cdot \mathbf{z}) d\mathbf{z}, \quad \vec{\lambda} \in (i\mathbb{R})^N, \quad \vec{\mu} \in \mathbb{W}_N, \quad (83)$$

and [19]

$$\Theta_N(t, \mathbf{y}) = \int_{(i\mathbb{R})^N} e^{t \sum_{j=1}^N \lambda_j^2/2} \psi_{-\vec{\lambda}}^{(N)}(\mathbf{y}) s_N(\vec{\lambda}) d\vec{\lambda}. \quad (84)$$

At time $t = T$, (81) gives the distribution

$$P_{N,T}^{\vec{\mu}}(T, \mathbf{y} | -\infty) = \frac{1}{J^{\vec{\mu}}(N, T)} \Theta_N(T, \mathbf{y}) \exp(-\vec{\mu} \cdot \mathbf{y}), \quad \mathbf{y} \in \mathbb{R}^N, \quad T > 0. \quad (85)$$

On the other hand, if we take the limit $T \rightarrow \infty$, (81) gives the distribution

$$\begin{aligned} P_N^{\vec{\mu}}(t, \mathbf{y} | -\infty) &\equiv \lim_{T \rightarrow \infty} P_{N,T}^{\vec{\mu}}(t, \mathbf{y} | -\infty) \\ &= e^{-|\vec{\mu}|^2 t/2} \Theta_N(t, \mathbf{y}) \psi_0^{(N)}(\mathbf{y}), \quad \mathbf{y} \in \mathbb{R}^N, \quad t > 0. \end{aligned} \quad (86)$$

The three-dimensional Bessel process is defined as the radial part of the three-dimensional BM and abbreviated to BES(3). The above results will be compared with the Imhof relation between BES(3) and the process called a meander and its multivariate generalizations discussed in [5, 10].

VII. SUMMARY AND CONCLUDING REMARKS

The vicious BM is obtained as a diffusion scaling limit of Fisher's vicious walk model [1, 5, 6]. It is an N -particle system of BMs in one dimension, whose positions are arranged in the order $x_1 < x_2 < \dots < x_N$, such that if and only if two neighboring Brownian particles collide with each other then they are pair annihilated, while they can enjoy free Brownian motions if they are all located separately from each other. In the present paper we have considered a system of N Brownian particles with the killing term

$$-V_N(\mathbf{x}) = - \sum_{j=1}^{N-1} e^{-(x_{j+1}-x_j)}. \quad (87)$$

That is, the interactions between neighboring Brownian particles are long-ranged and the risk to be pair annihilated exists always, which is expressed by a rapid decreasing function

(87) of the distance of the two particles $x_{j+1} - x_j$. We regard this system of mutually killing BMs as a generalized version of vicious BM, since the original vicious BM can be identified with the system of BMs with the killing term obtained by $-\lim_{\varepsilon \rightarrow 0, \varepsilon > 0} V_N(\mathbf{x}/\varepsilon)$.

Though the original vicious BM has only contact interactions, if we consider the system conditioned never to collide with each other, then we obtain a system of BMs with long-ranged interactions; the SDE is given by Eq.(8), in which between any pair of particles there acts a repulsive force proportional to the inverse of distance of the pair [8]. This N -particle process is equivalent to the eigenvalue process of an $N \times N$ Hermitian-matrix-valued BM introduced by Dyson in order to dynamically simulate the eigenvalue statistics of the Gaussian unitary ensemble (GUE) of random matrices (the Dyson model) [15–17].

As discussed in [45, 46], the equivalence between the eigenvalue process of Dyson and the noncolliding BM (the vicious BM conditioned never to collide) is the N -variate extension of the equivalence between BES(3) and the one-dimensional BM conditioned to stay positive. In this sense, the Dyson model can be regarded as a many-particle generalization of BES(3).

Apart from the equivalence between the BES(3) and the conditional BM to stay positive, the following equivalence is established. Let $M(t) = \max_{0 \leq s \leq t} B(s)$, $t \geq 0$, and define a process $Y(t) = 2M(t) - B(t)$, $t \geq 0$. Then $Y(t)$ is equivalent to BES(3), which is known as Pitman's ‘ $2M - X$ ’ theorem [47] (see also [40, 41, 48]). As a multivariate extension of Pitman's ‘ $2M - X$ ’ theorem, another construction of the Dyson model (the noncolliding BM) has been reported [25, 26, 28, 31, 32].

Matsumoto and Yor studied the stochastic process $Z_{\text{MY}}^\mu(t)$, $t \geq 0$ given by (74). We can see that

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0, \varepsilon > 0} \varepsilon Z_{\text{MY}}^\mu(t/\varepsilon^2) \\ &= \lim_{\varepsilon \rightarrow 0, \varepsilon > 0} \left[\varepsilon \log \left\{ \int_0^t e^{2B^\mu(s)/\varepsilon} ds \right\} - B^\mu(t) - \varepsilon \log \varepsilon^2 \right] \\ &= 2 \max_{0 \leq s \leq t} B^\mu(s) - B^\mu(t), \quad t \geq 0. \end{aligned} \tag{88}$$

Then, when $\mu = 0$, this $\varepsilon \rightarrow 0$ limit is equivalent to $Y(t)$ and thus with the BES(3). In this sense, the Matsumoto-Yor process is a generalization of the BES(3) [40, 41].

O’Connell [19] introduced an N -particle process $\mathbf{Z}^{\vec{\mu}}(t) = (Z_1^{\vec{\mu}}(t), \dots, Z_N^{\vec{\mu}}(t))$, $t \geq 0$, $\vec{\mu} \in \mathbb{R}^N$ as a multi-dimensional generalization of the Matsumoto-Yor process. Corresponding to the fact that the Matsumoto-Yor process is a generalization of the BES(3), the O’Connell process

is a generalization of the Dyson model. Actually, he showed that $\lim_{\varepsilon \rightarrow 0, \varepsilon > 0} \varepsilon \mathbf{Z}^0(t/\varepsilon^2), t \geq 0$ is equivalent to the Dyson model in the sense of an extension of Pitman's ' $2M - X$ ' theorem [19]. We pointed out that the BES(3), the original vicious BM, and the Dyson model (the noncolliding BM) can be regarded as *ultradiscretizations* [49] of the Matsumoto-Yor process, the BMs with the killing term in the same form as the quantum Toda lattice potential, and the O'Connell process, respectively.

In the present paper, we discussed another construction of the O'Connell process apart from the extension of Pitman's ' $2M - X$ ' theorem. In the special case with $\vec{\mu} = 0$, we have shown here that his process is given as a generalized version of vicious BM conditioned to survive forever. In order to demonstrate that the relation between the present generalized vicious BM and the O'Connell process is a multivariate generalization of the relation between a killing BM and the Matsumoto-Yor process, we showed in Sec.V that the Matsumoto-Yor process with $\mu = 0$ is obtained as a killing BM conditioned to survive forever.

We want to emphasize that the present analysis is indeed based on the idea of O'Connell to discuss interacting diffusive particle systems using the exact solutions of the quantum Toda lattices [19].

In Sec.I in the present paper, we listed up three fundamental properties of the noncolliding BM. They are all inherited by the O'Connell process in the extended form. (i) The Karlin-McGregor determinantal expression (4) of $q_N(t, \mathbf{y}|\mathbf{x})$, which is expanded by the Schur functions (32), is generalized by the integral formula (31). (ii) The harmonic transform [13] from q_N to p_N (6) by the harmonic function h_N given by the product of differences of variables (the Vandermonde determinant) (2) is now given by the formula (46) from Q_N to P_N . There h_N is replaced by an eigenfunction $\psi_0^{(N)}$ of the infinitesimal generator \mathcal{L}_N (the Hamiltonian \mathcal{H}_N of the quantum Toda lattice). (See [50–52] for harmonic transforms of one dimensional generalized diffusion processes.) (iii) Theorem 2 in Sec.IV gives the extended version of the Kolmogorov equation of (7).

There are a lot of future problems. In noncolliding diffusion processes, if we study the situations starting from “the all zero state” and observe particle distributions at an arbitrary time $0 < t < \infty$ in temporally homogeneous processes, and at the ending time $t = T$ in temporally inhomogeneous processes defined only in a finite time-interval $[0, T]$, we have obtained the eigenvalue distributions of random matrices in a variety of ensembles studied in random matrix theory [10]. In Sec.VI, we demonstrated that “the all $-\infty$ state” and

temporally inhomogeneous versions of processes will play important roles in the O’Connell process. The noncolliding diffusion process is determinantal, in the sense that for any finite initial configuration all multitime correlation functions are given by determinants associated with an integral kernel called the correlation kernel [53, 54]. It will be a challenging problem to clarify how matrix-structures (*i.e.* symmetries of systems) [16, 17] and solvability are inherited by the family of O’Connell processes.

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Appendix A: Proof of (71)

Let $J_0(z)$ be the Bessel function of the first kind of order 0,

$$J_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{(k!)^2}, \quad |z| < \infty. \quad (\text{A1})$$

Since the equality

$$K_\lambda(x)K_\lambda(y) = \frac{\pi}{2 \sin(\pi\lambda)} \int_{\log(y/x)}^{\infty} J_0(\sqrt{2xy \coth u - x^2 - y^2}) \sinh(u\lambda) du \quad (\text{A2})$$

holds for $x > 0, y > 0, |\operatorname{Re}\lambda| < 1/4$ [43], (69) is written as

$$\begin{aligned} Q^\mu(t, y|x) &= e^{-\mu^2 t/2 + \mu(x-y)} \frac{i}{2\pi^2} \int_{x-y}^{\infty} du J_0(\sqrt{2e^{-(x+y)} \cosh u - e^{-2x} - e^{-2y}}) \\ &\quad \times \int_{-i\infty}^{i\infty} d\lambda e^{\lambda^2 t/2} \lambda \sinh(u\lambda) \\ &= \frac{e^{-\mu^2 t/2 + \mu(x-y)}}{\sqrt{2\pi} t^{3/2}} \int_{x-y}^{\infty} u J_0(\sqrt{2e^{-(x+y)} \cosh u - e^{-2x} - e^{-2y}}) e^{-u^2/2t} du, \end{aligned} \quad (\text{A3})$$

where we have performed the integral of λ over $i\mathbb{R}$. This gives

$$\begin{aligned} \lim_{t \rightarrow \infty} \sqrt{\frac{\pi}{2}} t^{3/2} e^{\mu^2 t/2} Q^\mu(t, y|x) \\ = e^{\mu(x-y)} \frac{1}{2} \int_{x-y}^{\infty} u J_0(\sqrt{2e^{-(x+y)} \cosh u - e^{-2x} - e^{-2y}}) du. \end{aligned} \quad (\text{A4})$$

We find that the $\lambda \rightarrow 0$ limit of (A2) gives the equality

$$K_0(x)K_0(y) = \frac{1}{2} \int_{\log(y/x)}^{\infty} u J_0(\sqrt{2e^{-(x+y)} \cosh u - e^{-2x} - e^{-2y}}) du, \quad (\text{A5})$$

and then (A4) gives

$$\lim_{t \rightarrow \infty} \sqrt{\frac{\pi}{2}} t^{3/2} e^{\mu^2 t/2} Q^\mu(t, y|x) = e^{\mu(x-y)} K_0(e^{-x}) K_0(e^{-y}). \quad (\text{A6})$$

The asymptotics of $K_0(e^{-y})$ is known as [43]

$$K_0(e^{-y}) \simeq \begin{cases} \log(2/e^{-y}) \sim y & \text{as } y \rightarrow \infty \\ \sqrt{\pi/(2e^{-y})} \exp(-e^{-y}) \rightarrow 0 & \text{as } y \rightarrow -\infty. \end{cases} \quad (\text{A7})$$

Then if $\mu > 0$, $\int_{-\infty}^{\infty} K_0(e^{-y}) e^{-\mu y} dy < \infty$. Actually, for $\mu > 0$, this integral is the Mellin transformation of $K_0(z)$ and we obtain

$$\int_{-\infty}^{\infty} K_0(e^{-y}) e^{-\mu y} dy = \int_0^{\infty} K_0(z) z^{\mu-1} dz = 2^{\mu-2} (\Gamma(\mu/2))^2. \quad (\text{A8})$$

Therefore (71) is valid. ■

We note that, for the function

$$\theta_r(t) = \frac{i}{2\pi^2} \int_{-i\infty}^{i\infty} e^{\lambda^2 t/2} K_\lambda(r) \lambda \sin(\pi \lambda) d\lambda, \quad r > 0, \quad (\text{A9})$$

Yor gave the following expression (see Eq.(6.b”) on page 43 of [44]),

$$\theta_r(t) = \frac{r}{(2\pi^3 t)^{1/2}} e^{\pi^2/2t} \int_0^{\infty} e^{-\eta^2/2t} e^{-r \cosh \eta} (\sinh \eta) \sin\left(\frac{\pi \eta}{t}\right) d\eta, \quad r > 0. \quad (\text{A10})$$

Using this expression, Matsumoto and Yor reported the asymptotics (see Eq.(2.11) in [41]),

$$\lim_{t \rightarrow \infty} \sqrt{2\pi t^3} \theta_r(t) = K_0(r), \quad r > 0. \quad (\text{A11})$$

Since the equality

$$Q^\mu(t, y|x) = e^{-\mu^2 t/2 + \mu(x-y)} \int_0^{\infty} \exp\left\{-\frac{s}{2} - \frac{1}{2s}(e^{-2x} + e^{-2y})\right\} \theta_{e^{-(x+y)}/s}(t) \frac{ds}{s} \quad (\text{A12})$$

is established, the limit (A6) can be concluded also from (A11).

Appendix B: $N = 2$ case of the O'Connell process

By the equations (25), (28) and (63), we obtain

$$Q_2(t, \mathbf{y}|\mathbf{x}) = \frac{1}{2\pi^3} \int_{-i\infty}^{i\infty} d\lambda_1 \int_{-i\infty}^{i\infty} d\lambda_2 e^{(\lambda_1^2 + \lambda_2^2)t/2} e^{(\lambda_1 + \lambda_2)\{(x_1 + x_2) - (y_1 + y_2)\}/2} \\ \times K_{\lambda_1 - \lambda_2}(2e^{-(x_2 - x_1)/2}) K_{\lambda_1 - \lambda_2}(2e^{-(y_2 - y_1)/2}) (\lambda_1 - \lambda_2) \sin\{\pi(\lambda_1 - \lambda_2)\}, \quad (\text{B1})$$

$\mathbf{x}, \mathbf{y} \in \mathbb{R}^2, t \geq 0$. If we change the integral variables $(\lambda_1, \lambda_2) \mapsto (\lambda, \nu)$ by $\lambda = \lambda_1 - \lambda_2, \nu = \lambda_1 + \lambda_2$, we can calculate the integral with respect to ν . The result is expressed by using the transition probability density Q^μ of the Matsumoto-Yor process (69) with $\mu = 0$ as

$$Q_2(t, \mathbf{y}|\mathbf{x}) = p(2t, y_1 + y_2 | x_1 + x_2) Q^0\left(\frac{t}{2}, \frac{y_2 - y_1}{2} - \log 2 \middle| \frac{x_2 - x_1}{2} - \log 2\right), \quad (\text{B2})$$

where $p(t, y|x) = e^{-(y-x)^2/2t}/\sqrt{2\pi t}$. Therefore, from the long-term asymptotics (A6) of Q^μ , we can obtain the long-term asymptotics of Q_2 as

$$Q_2(t, \mathbf{y}|\mathbf{x}) \simeq \frac{2}{\pi t^2} K_0(2e^{-(x_2 - x_1)/2}) K_0(2e^{-(y_2 - y_1)/2}) \\ = \frac{t^{-2}}{2\pi} \psi_0^{(2)}(\mathbf{x}) \psi_0^{(2)}(\mathbf{y}) \quad \text{as } t \rightarrow \infty, \quad (\text{B3})$$

which coincides with the $N = 2$ case of (34) in Lemma 1.

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